A Modified Algorithm For Solving Split Feasibility Problem For Countable Family Of Multivalued $\kappa-$ Demi-Contractive Mappings And Total Asymptotically Strict Pseudo-Contractive Mapping.

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ABSTRACT

The purpose of this article is to study the split feasibility problem for a countable family of multi-valued κ -demicontractive mappings and total asymptotically strict pseudo-contractive mapping in infinite dimensional Hilbert spaces. In order to solve the problem, we constructed a modified Ishikawa's type of algorithm. Under mild conditions, we established convergence theorem using the sequences of the algorithm. The main results presented in the paper improve and extend the recent result of Chang et al. [Applied Mathematics and Computation 219(2013) 10416-10424] , J.N.Ezeora and R.C.Ogbonna[Matematicki Vesnik 70,3(2018),233-242].

Key words and phrases: Split feasibilty problem; Countable family; Multi- valued mappings; demicontractive mapping; Contractive mapping; Pseudocontractive mapping; Strictly pseudocontractive mapping; asymptotically strict pseudocontractive mapping; total asymptotically strict pseudocontractive mapping; Hilbert space.

1.INTRODUCTION

Generally, problems involving nonlinear equations have no known methods to obtain closed form solutions for them. An iterative algorithm is a procedure used for solving a problem or performing a computation. It is usually the only way to solve problems involving nonlinear equations by finding an approximate solution to the given problem. There is no universal iterative algorithm applicable to all kinds of problems involving nonlinear equations. Several iterative algorithms have been deployed to approximate solutions, assuming existence. Recently, iterative algorithms have been utilized by researchers to solve split feasibility problems.

Based on the aforementioned results and some other result in this direction, the need to construct a modified iterative algorithm for solving split feasibility problem involving nonlinear equations arises.

In this work, H will be used to denote a real Hilbert space and K will denote, a subset of H. A mapping $T: K \to K$ is said to be $(\kappa, \{\mu_n\}, \{\varepsilon_n\}, \phi)$-total asymptotically strict pseudo contractive, if there exists a constant $\kappa \in [0,1)$ and sequences $\{\mu_n\} \subset [0, \infty), \{\varepsilon_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, and a continuous and strictly increasing function $\phi: [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $n \geq 1, x, y \in K$,

$$\|T^n x - T^n y\|^2 \leq \| x - y \|^2 + k\|x - y - (T^n x - T^n y)\|^2 + \mu_n \phi(\| x - y \|) + \varepsilon_n \quad (1)$$

Let (X, d) be a metric space and $CB(X)$ be the family of all closed and bounded subsets of X. Let $\mathcal{H}$ denote the Hausdorff metric induced by the metric d, then for all $A, B \in CB(X)$,

$$\mathcal{H}(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\} \quad (2)$$

where $d(a, B) := \inf_{b \in B} d(a, b)$.

Let $T: D(T) \subseteq H \to CB(H)$ be a multi-valued mapping, a point $x \in D(T)$ is called a fixed point of T if $x \in Tx$. A multi-valued mapping T is said to be L-Lipschitzian if there exists $L > 0$ such that

$$\mathcal{H}(Tx, Ty) \leq L \| x - y \|, x, y \in D(T). \quad (3)$$

In (3), if $L \in (0,1)$, then T is called contraction while T is called nonexpansive if $L = 1$. T is said to be

(i) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$\mathcal{H}(Tx, Ty) \leq \| x - y \|, \forall x \in D(T), y \in F(T)$,

(ii) κ-strictly pseudocontractive (see e.g., [12,13]) if there exists $\kappa \in [0,1)$ such that $\forall x, y \in D(T)$,

$\mathcal{H}^2(Tx, Ty) \leq \| x - y \|^2 + k \| x - u - (y - v) \|^2, \forall u \in Tx, v \in Ty$.

(iii) κ - demicontractive if $F(T) \neq \emptyset$ and for $\kappa \in [0,1)$ we have

$\mathcal{H}^2(Tx - p) \leq \| x - p \|^2 + k \| x - Tx \|^2, \forall x \in D(T)$ and $\forall p \in F(T)$

Clearly,the following inclusions hold for the above classes of mappings; nonexpansive $\subset$ quasi - nonexpansive $\subset$ κ-strictly pseudocontractive $\subset$ κ-demicontractive.

Example 1.1. [15] Let $H = \mathbb{R}$ and $C = [-1,1]$. Define $T: C \to C$ by

$$Tx = \begin{cases} \frac{2}{3} x\mathrm{Sin}\left(\frac{1}{x}\right), x \neq 0 \\ 0, x = 0 \end{cases} \quad (4)$$

Clearly, $F(T) = \{0\}$, for $x \in C$, we have

$$|Tx - 0|^2 = \left|\frac{2}{3}x\mathrm{Sin}\left(\frac{1}{x}\right)\right|^2 \le \left|\frac{2}{3}x\right|^2 \le |x|^2 \le |x-0|^2 + \kappa|x - Tx|^2 \forall \kappa \in [0,1)$$

Thus T is κ-demicontractivefor $\kappa \in [0,1)$

T is not κ - strictly pseudocontractive, choose $x = \frac{2}{\pi}$ and y $= \frac{2}{3\pi}$, then

$$|Tx - Ty|^2 > |x-y|^2 + \kappa|x - y - (Tx - Ty)|^2$$

Hence, T is not κ - strictly pseudocontractive mapping for $\kappa \in [0,1)$

Example 1.2. ([11]) Let $H = \mathbb{R}$ with the usual metric. Define $T: H \to 2^H$ by

$$Tx = \begin{cases} \left[-3x, -\frac{5x}{2}\right], x \in [0,\infty) \\ \left[-\frac{5x}{2}, -3x\right], x \in (-\infty, 0] \end{cases} \quad (5)$$

We have that, $F(T) = \{0\}$ and T is a multivalued demi-contractive mapping which is not quasi-nonexpansive . For each, $x \in (-\infty, 0) \cup (0, \infty)$, we have

$$|Tx - T0|^2 = |-3x - 0|^2 = 9|x-0|^2$$

which implies that T is not quasi-nonexpansive.

Also, we have that

using above,

$$|x - Tx|^2 = \left|x - \left(-\frac{5x}{2}\right)\right|^2 = \frac{49}{4}|x-0|^2$$

$$|Tx - T0|^2 = 9|x-0|^2 = |x-0|^2 + 8|x-0|^2 = |x-0|^2 + \frac{32}{49}|x - Tx|^2$$

Therefore, T is a demi-contractive mapping with constant $\kappa = \frac{32}{49}$

In 1994,Censor and Elfving [4] introduced, in finite dimensional Hilbert spaces, the split feasibility problem for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. It is now known that split feasibility problems can be used in various disciplines, such as image restoration, computer tomograph and radiation therapy treatment planning (see [2,3,5,6]).

Let H_1 and H_2 be two real Hilbert spaces, K and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The split feasibility problem is formulated as follows; find a point $q \in H_1$ such that

$$q \in K \text{ and } Aq \in Q, \quad (6)$$

where $A: H_1 \to H_2$ is a bounded linear operator. If (6) has solution, it can be shown that $x \in K$ solves (6) if and only if it solves the following fixed point equation:

$$x = P_K\left(\left(I - \gamma A^*(I - P_Q)A\right)x\right), x \in K, \qquad (7)$$

where P_K and P_Q are the projections onto K and Q, respectively, γ is a positive constant, and A^* denotes the adjoint of A. When K and Q in (6) are the sets of fixed points of two nonlinear mappings, and K and Q are nonempty closed convex subsets of H_1 and H_2, respectively, then the split feasibility problem (6) is also called split common fixed point problem or multiple-set split feasibility problem (see [9, 20]).

Split common fixed point problems for nonlinear mappings in the setting of two Hilbert spaces have been studied by many authors; (see, for instance, [7, 8, 10, 16, 17, 18]).

In 2013, Chang et.al [9] obtained strong and weak convergence results for multiple set split feasibility problems for a family of multi-valued mappings and a single valued nonlinear mapping in Hilbert spaces.

Recently, Ezeora and Ogbonna [14] study and obtained strong and weak convergence results for multiple set split feasibility problems of a countable family of multi-valued κ-strictly pseudo-contractive mapping and total asymptotically strict pseudo-contractive mapping in infinite dimensional Hilbert spaces.

Motivated by the results of Chang et. al,Ezeora and Ogbonna, we introduce and study multiple-set split feasibility problem for a countable family of multi-valued κ - demicontractive mapping and total asymptotically strict pseudo-contractive mapping in infinite dimensional Hilbert spaces.

2. PRELIMINARIES

In the sequel, we shall denote by $\rightharpoonup$ and $\to$, the weak and strong convergence of a sequence $\{x_n\}$, respectively.

Definition 2.1. A multi-valued mapping $T: D(T) \subseteq H_1 \to CB(H_1)$ is said to be demi-closed at origin if for any sequence $\{x_n\} \subset H_1$ with $x_n \rightharpoonup q$ and $d(x_n, Tx_n) \to 0$, we have $q \in Tq$.

Definition 2.2. A normed linear space, X is said to satisfy opial's condition if, for any sequence, $\{x_n\}$ such that $x_n \rightharpoonup p$, we have

$$\liminf_{n\to\infty}\|x_n - p\| < \liminf_{n\to\infty}\|x_n - q\| \forall q \in X \text{ with } q \neq p$$

A multi-valued mapping $T: D(T) \subseteq H_1 \to CB(H_1)$ is said to be hemi-compact if, for any sequence $\{x_n\}$ in H_1 such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in H_1$.

Definition 2.3. Let $\{x_n\}$ be a sequence in H. A point $x^* \in H$ is called a weak cluster point of the sequence $\{x_n\}$ if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$, as $j \to \infty$.

Lemma 2.4. (see [11]) Let K be a nonempty set of real Hilbert space H and let $T: K \to CB(K)$ be a multivalued k - demi-contractive mapping. Assume that for every $p \in F(T), Tp = \{p\}$. Then, there exists $L > 0$ such that

$\mathcal{H}(Tx,Tp) \leq L \parallel x-p \parallel \forall x \in K, p \in F(T)$.

Lemma 2.5. (see [9] Lemma 2.3) Let $T: H \to H$ be a uniformly L-Lipschitzian continuous and $(k,\{\mu_n\},\{\varepsilon_n\},\phi)$ total asymptotically strict pseudo contractive mapping, then T is demi-closed at the origin.

Lemma 2.6. (see [12]) Let $\{x_i\}_{i=1}^{\infty} \subset H$ and $\alpha_i \in (0,1), i = 1,2, \ldots$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. If $\{x_i\}_{i=1}^{\infty}$ is bounded, then

$$\|\sum_{i=1}^{\infty} \alpha_i x_i\|^2 = \sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 - \sum_{i,j=1, i \neq j}^{\infty} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.7. (see [19]) Let $\{a_n\},\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying $a_{n+1} \leq (1+\delta_n)a_n + b_n \forall n \geq 1$. If $\sum_{i=1}^{\infty} \delta_n < \infty$ and $\sum_{i=1}^{\infty} b_n < \infty$, then the $\lim_{n\to\infty} a_n$ exists.

3. MAIN RESULT

For solving the multiple-set split feasibility problem, we assume that the following conditions are satisfied:

(1) H_1 and H_2 are two real Hilbert spaces, $A: H_1 \to H_2$ is a bounded linear operator and $A^*: H_2 \to H_1$ is the adjoint of A.

(2) $T_i: H_1 \to H_1, i = 1,2, \ldots$ is a countable family of multi-valued $\kappa_i -$ demicontractive mappings and for each $i \geq 1, (I - T_i)$ is demi-closed at the origin.

(3) $\mathrm{T}: \mathrm{H}_2 \to \mathrm{H}_2$ is a uniformly L-Lipschitzian continuous and $(k,\{\mu_n\},\{\varepsilon_n\},\phi)$ - total asymptotically strict pseudo-contractive mapping satisfying the following conditions:

(i) $\sum_{i=1}^{\infty} \mu_n < \infty; \sum_{i=1}^{\infty} \varepsilon_n < \infty$:

(ii) there exist constants $M > 0, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda^2, \forall \lambda \geq M$:

(4) $K := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $Q := F(T) \neq \emptyset$.

(5) For each $q \in K, T_i q = \{q\}$ for each $i \geq 1$.

The set of solution of the multiple set split feasibility problem will be denoted by

$\mathcal{F}$. i.e., $\mathcal{F} = \{x \in K: Ax \in Q\} = K \cap A^{-1}(Q)$.

Theorem 3.1. Let $H_1, H_2, A, A^*, T_i, T, K, Q, \kappa, \{\mu_n\},\{\varepsilon_n\}$, ϕ and L satisfy conditions (1)-(5) above. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_1 \in H_1 \text{ chosen arbitrary} \\ x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} w_{i,n}, w_{i,n} \in T_i y_n \\ y_n = x_n + \beta A^*(T^n - I)Ax_n, \forall n \geq 1 \end{cases} \tag{8}$$

where $\{\alpha_{i,n}\} \subset (k,1)$ with $k = \sup_{i>1} k_i \in [0,1)$ and $\beta > 0$ satisfy the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{i,n} = 1$, for each $n \geq 1$, (b) for each $i \geq 1$, $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} \geq 0$,

(c) $\beta \in \left(0, \frac{1-k}{\|A\|^2}\right)$.

If $\mathcal{F}$ is nonempty, then

(A) both $\{x_n\}$ and $\{y_n\}$ converge weakly to some point $q \in \mathcal{F}$.

(B) In addition, if there exists some positive integer m such that $T_m \in \{T_i\}_{i=1}^{\infty}$ is hemi-compact, then both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \mathcal{F}$.

Proof. We observe that by condition (3)(ii), and the fact that ϕ is a continuous and strictly increasing function, there exist constants $M > 0, M^* > 0$ such that

$$\phi(\lambda) \leq \phi(M) + M^*\lambda^2, \forall \lambda \geq 0. \quad (9)$$

To establish conclusion [A], we divide the proof into six steps.

Step 1

We prove that all the sequences $\{x_n\}, \{y_n\}$ and $\{w_{i,n}\}$ are bounded. and for each $q \in \mathcal{F}$, we also show that $\lim_{n\to\infty} \|x_n - q\| = \lim_{n\to\infty} \|y_n - q\|$.

For any given $q \in \mathcal{F}$, we have $q \in K$ and $Aq \in Q := F(T)$.

By the assumption that for each $i \geq 1, T_i$ is a multi-valued κ - demicontractive mapping, therefore the fixed point set $F(T_i)$ is closed, and so is $K := \cap_{i=1}^{\infty} F(T_i)$.

From equation 3.1, condition (5) and using Lemma 2.6, we have that for each $n \geq 1$ and $q \in \mathcal{F}$,

$$\|x_{n+1} - q\|^2 = \left\|\alpha_{0,n}(y_n - q) + \sum_{i=1}^{\infty} \alpha_{i,n}(w_{i,n} - q)\right\|^2 \quad (10)$$

$$= \alpha_{0,n}\|(y_n - q)\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n}\|w_{i,n} - q\|^2 - \sum_{i=1}^{\infty} \alpha_{i,n}\alpha_{0,n}\|y_n - w_{i,n}\|^2$$

$$- \sum_{i,j=1,i\neq j}^{\infty} \alpha_{i,n}\alpha_{j,n}\|w_{i,n} - w_{j,n}\|^2$$

$$\leq \alpha_{0,n}\|(y_n - q)\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n}\|w_{i,n} - q\|^2 - \sum_{i=1}^{\infty} \alpha_{i,n}\alpha_{0,n}\|y_n - w_{i,n}\|^2$$

$$\leq \alpha_{0,n}\|(y_n - q)\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n}D(T_i y_n - T_i q)^2 - \sum_{i=1}^{\infty} \alpha_{i,n}\alpha_{0,n}\|y_n - w_{i,n}\|^2$$

$$\leq \alpha_{0,n}\|(y_n - q)\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n}\left(\|y_n - q\|^2 + k\|y_n - T_i y_n\|^2\right) - \sum_{i=1}^{\infty} \alpha_{i,n}\alpha_{0,n}\|y_n - w_{i,n}\|^2$$

$$\leq \alpha_{0,n}\|(y_n - q)\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n}\left(\|y_n - q\|^2 + k\|y_n - w_{i,n}\|^2\right) - \sum_{i=1}^{\infty} \alpha_{i,n}\alpha_{0,n}\|y_n - w_{i,n}\|^2$$

$$= \sum_{i=0}^{\infty} \alpha_{i,n}\|y_n - q\|^2 - \sum_{i=1}^{\infty} \alpha_{i,n}(\alpha_{0,n} - k)\|y_n - w_{i,n}\|^2 \quad (11)$$

$$= \|y_n - q\|^2 - \sum_{i=1}^{\infty} \alpha_{i,n}(\alpha_{0,n} - k)\|y_n - w_{i,n}\|^2 \quad (12)$$

$$= \|y_n - q\|^2 - (1 - \alpha_{0,n})(\alpha_{0,n} - k)\|y_n - w_{i,n}\|^2 \quad (13)$$

$$\leq \|y_n - q\|^2$$

$$\|y_n - q\|^2 = \|x_n - q\|^2 + 2\beta\langle x_n - q, A^*(T^n - I)Ax_n\rangle + \beta^2\|A^*(T^n - I)Ax_n\|^2. \quad (14)$$

Notice that

$$\beta^2\|A^*(T^n - I)Ax_n\|^2 \leq \beta^2\|A\|^2\|(T^n - I)Ax_n\|^2. \quad (15)$$

Since $q \in K, Aq \in F(T)$. T is a $(k, \{\mu_n\}, \{\varepsilon_n\}, \emptyset)$ - total asymptotically strict pseudo contractive mapping, we have

$$
\begin{aligned}
\langle x_n - q, A^*(T^n - I)Ax_n\rangle &= \langle A(x_n - q), (T^n - I)Ax_n\rangle \\
&= \langle A(x_n - q) + (T^n - I)Ax_n - (T^n - I)Ax_n, (T^n - I)Ax_n\rangle \\
&= \langle (T^nAx_n - Aq), (T^n - I)Ax_n\rangle - \|(T^n - I)Ax_n\|^2 \\
&= \frac{1}{2}\{\|T^nAx_n - Aq\|^2 + \|(T^n - I)Ax_n\|^2 - \|Ax_n - Aq\|^2 - \|(T^n - I)Ax_n\|^2\} \\
&\le \frac{1}{2}\{\|Ax_n - Aq\|^2 + \kappa\|(T^n - I)Ax_n\|^2 + \mu_n\phi(\|Ax_n - Aq\|) + \varepsilon_n\} \\
&+ \frac{1}{2}\{\|(T^n - I)Ax_n\|^2 - \|Ax_n - Aq\|^2\} - \|(T^n - I)Ax_n\|^2 \\
&= \frac{\kappa - 1}{2}\|(T^n - I)Ax_n\|^2 + \frac{1}{2}\{\mu_n\phi(\|Ax_n - Aq\|) + \varepsilon_n\} \\
&\le \frac{k - 1}{2}\|(T^n - I)Ax_n\|^2 + \frac{\mu_n}{2}\{M^* \| A \|^2 \|x_n - q\|^2 + \phi(M)\} + \frac{1}{2}\varepsilon_n (\text{by } 9). \qquad (16)
\end{aligned}
$$

Using inequalities (15) and (16) in equation (14) , we obtain

$$
\|y_n - q\|^2 \le (1 + \beta\mu_n M^* \| A \|^2)\|x_n - q\|^2 - \beta(1 - k - \beta \| A \|^2)\|(T^n - I)Ax_n\|^2 +
$$

$$
\beta\{\mu_n\phi(M) + \varepsilon_n\} \qquad (17)
$$

By condition (c), $(1 - k - \beta \| A \|^2) > 0$, therefore we have

$$
\|y_n - q\|^2 \le (1 + \beta\mu_n M^* \| A \|^2)\|x_n - q\|^2 + \beta\{\mu_n\phi(M) + \varepsilon_n\}. \qquad (18)
$$

substituting inequality (18) into equation (10) , we have

$$
a_{n+1} \le (1 + \delta_n)a_n + b_n, \qquad (19)
$$

where $a_n = \|x_n - q\|^2, \delta_n = \beta\mu_n M^* \| A \|^2$ and $b_n = \beta\{\mu_n\phi(M) + \varepsilon_n\}$. Applying condition (3) (i), we get that $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. So, from Lemma 2.7, we get $\lim_{n\to\infty}\|x_n - q\|$ exists. From equation (10) and inequality(18), we know that $\lim_{n\to\infty}\|y_n - q\|$ exists. Also, $\lim_{n\to\infty}\|x_n - q\| = \lim_{n\to\infty}\|y_n - q\| \forall q \in \mathcal{F}$. Hence, $\{x_n\}$ and $\{y_n\}$ are bounded. Furthermore, Since for each i, T_i is a multi-valued k_i - demicontractive, we have from Lemma 2.4 that

$$
\|w_{i,n} - q\| \le \left(\mathcal{H}(T_i y_n, T_i q)\right) \le \frac{1+\sqrt{k}}{1-\sqrt{k}}\|y_n - q\| \le LL_0 = M_0 \qquad (20)
$$

So, from (20), we get that $\{w_{i,n}\}$ is also bounded.

Step 2

Now we prove that for any $i \ge 1$, the following conditions hold:

$$
\lim_{n\to\infty} d(y_n, T_i y_n) = 0, \text{ and } \lim_{n\to\infty}\|T^nAx_n - Ax_n\| = 0. \qquad (21)
$$

For any $q \in \mathcal{F}$, it follows from (),(18(and Lemma 2.6, that for any $n \ge 1$ we have

$$\|x_{n+1}-q\|^2 \le (1+\delta_n)\|x_n-q\| + b_n - \beta(1-k-\beta\parallel A\parallel^2)\|(T^n-I)Ax_n\|^2 + k\sum_{i=1}^{\infty}\alpha_{i,n}\|y_n-w_{i,n}\|^2 - \sum_{i=1}^{\infty}\alpha_{i,n}\alpha_{0,n}\|y_n-w_{i,n}\|^2 \quad (22)$$

So,

$$\beta(1-k-\beta\parallel A\parallel^2)\|(T^n-I)Ax_n\|^2 - k\sum_{i=1}^{\infty}\alpha_{i,n}\|y_n-w_{i,n}\|^2 + \sum_{i=1}^{\infty}\alpha_{i,n}\alpha_{0,n}\|y_n-w_{i,n}\|^2 \quad (23)$$
$$\le (1+\delta_n)\|x_n-q\| + b_n - \|x_{n+1}-q\| \to 0 \text{ as } n\to\infty$$

$$\beta(1-k-\beta\parallel A\parallel^2)\|(T^n-I)Ax_n\|^2 + (1-\alpha_{0,n})(\alpha_{0,n}-k)\|y_n-w_{i,n}\|^2 \le (1+\delta_n)\|x_n-q\| + b_n - \|x_{n+1}-q\| \to 0 \text{ as } n\to\infty$$

By conditions (b) and (c), and conclusion (23), we have

$$\lim_{n\to\infty}\|(T^n-I)Ax_n\| = 0$$
$$\lim_{n\to\infty}\|y_n-w_{i,n}\| = 0\,\forall w_{i,n}\in T_i y_n. \quad (24)$$

Hence, we have

$$\lim_{n\to\infty} d(y_n, T_i y_n) \le \lim_{n\to\infty}\|(y_n-w_{i,n})\| = 0 \quad (25)$$

Step 3

We prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

From conclusion(25), we know that $\|w_{i,n}-y_n\|\to 0$ and $\|y_n-x_n\| = \beta\|A^*(T^n-I)Ax_n\|\to 0, n\to\infty$.
Using convexity of $\|\cdot\|^2$, we have from (8)

$$\|x_{n+1}-x_n\|^2 = \left\|\alpha_{0,n}(y_n-x_n) + \sum_{i=1}^{\infty}\alpha_{i,n}(w_{i,n}-x_n)\right\|^2$$
$$= \left\|\alpha_{0,n}\beta(A^*(T^n-I)Ax_n) + \sum_{i=1}^{\infty}\alpha_{i,n}(w_{i,n}-x_n)\right\|^2$$
$$\le \alpha_{0,n}\|\beta(A^*(T^n-I)Ax_n)\|^2 + \sum_{i=1}^{\infty}\alpha_{i,n}(\|w_{i,n}-y_n\| + \|y_n-x_n\|)^2. \quad (26)$$

Hence from(26), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (27)$$

Step 4

We prove that $\lim_{n\to\infty}\|TAx_n-Ax_n\|=0$.

Since T is uniformly L - Lipschitzian continuous, (24) and (27) above hold, we obtain

$$\|TAx_n-Ax_n\|\le\|TAx_n-T^{n+1}Ax_n\|+\|T^{n+1}Ax_n-T^{n+1}Ax_{n+1}\|+\|T^{n+1}Ax_{n+1}-Ax_{n+1}\|+\|Ax_{n+1}-Ax_n\|$$
$$\le L\|Ax_n-T^nAx_n\|+(L+1)\parallel A\parallel\|x_{n+1}-x_n\|+\|T^{n+1}Ax_{n+1}-Ax_{n+1}\|\to 0 \text{ as } n\to\infty. \qquad (28)$$

Step 5

We prove that every weak-cluster point of the sequence $\{x_n\}$ and $\{y_n\}$, $q\in\mathcal{F}$.

Since $\{y_n\}$ is bounded and H_1 is reflexive, there exists a subsequence $\{y_{n_k}\}\subset\{y_n\}$ such that $y_{n_k}\rightharpoonup q\in H_1$.

Hence from step (2), $\lim_{k\to\infty}d\left(y_{n_k},T_iy_{n_k}\right)=0$.

Using the fact that $(I-T_i)$ is demi- closed at the origin, we get that $q\in F(T_i)$. Since $i\ge 1$ is arbitrary, we have $q\in K:=\cap_{i=1}^{\infty}F(T_i)$. On the other hand, it follows from (8) and (24) that

$$x_{n_k}=y_{n_k}-\beta A^*(T^{n_k}-I)Ax_{n_k}\rightharpoonup q. \qquad (29)$$

Since A is a bounded linear operator, we obtain from 3.22 that $Ax_{n_k}\rightharpoonup Aq$. Notice that by step (4), we have $\lim_{k\to\infty}\|TAx_{n_k}-Ax_{n_k}\|=0$

and by Lemma 2.5, $I-T$ is demi-closed at origin and so, $Aq\in F(T)=Q$. Hence, $q\in\mathcal{F}$.

Step 6

We prove that $x_n\rightharpoonup q$ and $y_n\rightharpoonup q$.

Suppose that there exists some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to p$ with $p\ne q$.

Using similar arguments as the ones above, we can prove that $p\in\mathcal{F}$. Hence from the conclusions of step **1** and the Opial's property of Hilbert space, we have

$$\begin{aligned}\liminf_{k\to\infty}\|x_{n_k}-q\| &< \liminf_{k\to\infty}\|x_{n_k}-p\| \\ &= \lim_{n\to\infty}\|x_n-p\| \\ &= \lim_{k\to\infty}\|x_{n_k}-p\| \\ &< \liminf_{k\to\infty}\|x_{n_k}-q\| \\ &= \lim_{n\to\infty}\|x_n-q\| \\ &= \liminf_{k\to\infty}\|x_{n_k}-q\|. \end{aligned} \tag{30}$$

This is a contradiction. Therefore, $x_n \rightharpoonup q$. By using (8), we have $y_n = x_n + \beta A^*(T^n - I)Ax_n \rightharpoonup q$. This completes the proof of conclusion (A). Next, we prove conclusion (B).

Without loss of generality, we may assume that T_1 is hemi-compact. From step (2), we have that $d(y_n, T_1 y_n) \to 0$ as $n \to \infty$.

Therefore, there exists a subsequence of $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \to t \in H_1$. Since $y_{n_k} \rightharpoonup q$, we have $q = t$ and so $y_{n_k} \to q \in \mathcal{F}$. By virtue of conclusions of step 1, we have

$$\lim_{n\to\infty}\|x_n - q\| = \lim_{n\to\infty}\|y_n - q\| = 0.$$

That is, $\{y_n\}$ and $\{x_n\}$ both converge strongly to the point $q \in \mathcal{F}$. This completes the proof.

Lemma 3.1. Let H be a real Hilbert space. There hold the following identity

$$\| tx + (1-t)y \|^2 = t\| x \|^2 + (1-t)\| y \|^2 - t(1-t)\| x-y \|^2 \quad \forall t \in [0,1], x, y \in H. \tag{31}$$

Remark 3.1. If $\{T_i\}$ is a family of single-valued strictly demicontractive self mappings of H_1, then Theorem 3.1 still holds. In fact we have the following result.

Theorem 3.2. Let H_1 and H_2 be two real Hilbert spaces and $\{T_i\}_{i=1}^{\infty}: H_1 \to H_1$ be a family of single-valued strictly demicontractive mappings, and for each $i \geq 1, I - T_i$ is demi-closed at 0. Let $H_1, H_2, A, A^*, T, K, Q, \kappa, \{\mu_n\}, \{\varepsilon_n\}, \phi$, and L satisfy same conditions as in Theorem 3.1 and $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_1 \in H_1 \text{ chosen arbitrary} \\ x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} T_i y_n \\ y_n = x_n + \beta A^*(T^n - I)Ax_n, \ \forall n \geq 1 \end{cases} \tag{32}$$

If $\mathcal{F} \neq \emptyset$, then the conclusions of Theorem 3.1 still hold.

Proof. Using Lemma 3.1, one can easily get that $\{x_n\}, \{y_n\}$ and $\{T_i y_n\}$ are all bounded. The rest of the proof follow as in the proof of Theorem 3.1.

4. APPLICATION

Let H be a real Hilbert space, $\{T_i\}: H \to H$ be a countable family of nonexpansive mapping with $\mathcal{F} := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$; and $T: H \to H$ be a nonexpansive mapping. The hierarchical variational inequality problem for family countable of nonexpansive mapping $\{T_i\}$ with respect to the nonexpansive mapping T is to find an $x^* \in \mathcal{F}$ such that

$$\langle x^* - Tx^*, x^* - x\rangle \leq 0 \; \forall x \in \mathcal{F}. \tag{33}$$

It is known that (33) is equivalent to the following fixed point problem:

$$\text{find } x^* \in \mathcal{F} \text{ such that } x^* = P_{\mathcal{F}}(Tx^*), \tag{34}$$

where $P_{\mathcal{F}}$ is the metric projection from H onto F. Letting $K = \mathcal{F}$ and $Q = F(P_{\mathcal{F}})$ (the fixed point set of $P_{\mathcal{F}}$) and $A = I$ (the identity mapping on H), then problem (34) is equivalent to the following multi-set split feasibility problem:

$$\text{find } x^* \in K \text{ such that } Ax^* \in Q. \tag{35}$$

Theorem 4.2. Let $H, \{T_i\}, K and Q$ be the same as above. Let $\{x_n\}, \{y_n\}$ be the sequence defined by

$$\begin{cases} x_1 \in H_1 \text{ chosen arbitrary} \\ x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} T_i y_n, \\ y_n = x_n + \beta(T^n - I)x_n, \; \forall n \geq 1. \end{cases} \tag{36}$$

where $\{\alpha_n\} \subset (0,1)$ and $\beta > 0$ satisfy the following conditions:

(i) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$;

(ii) $\beta \in (0,1)$.

If $K \cap Q \neq \emptyset$, then $\{x_n\}$ converges weakly to a solution of the hierarchical variational inequality problem (33). In addition, if the mapping G is semi-compact, then both $\{x_n\}$ and $\{y_n\}$ converge strongly to a solution of the hierarchical variational inequality problem (33).

Proof. Since for each $i \geq 1, F(T_i) \neq \emptyset$ is nonexpansive, it is quasi-nonexpansive. Every quasi-nonexpansive mapping is strictly pseudo-contractive mapping and every strictly pseudo-contractive mapping is demi-contractive. Furthermore, since T is nonexpansive, it is uniformly L-Lipschitzan continuous and $(\kappa, \{\mu_n\}, \{\varepsilon_n\}, \phi)$-total asymptotically strict pseudocontractive with $L = 1, \mu_n \equiv 0, \varepsilon_n \equiv 0$, and $\phi = 0$. Therefore, all conditions of Theorem 3.1 are satisfied. Hence, the conclusions of Theorem 4.2 follow from Theorem 3.1.

Remark 4.1. A look at the proof of Theorem 3.1 shows that the proof carries over to the case of multi valued quasi nonexpansive maps and k-strictly pseudo-contractive maps. Consequently, Theorem 3.1 is an improvement and extension of Theorem 3.1 of Chang et al [9], Theorem 3.1 of Ezeora and Ogbonna [14] and other important results in this direction of research.

REFERENCES

[1] Boonchari, D. and Saejung, S.,(2009). Weak and strong convergence theorems of an implicit iteration for a countable family of continuous pseudocontractive mappings. J. Comput. App. Math., 33 , 1108-1116

[2] Byrne, C.,(2002). Iterative oblique projection onto convex subsets and the split feasibility problems. Inverse Problem, 18 , 441-453

[3] Censor, Y., Bortfeld, T., Martin, B. and Trofimov, A.,(2006). A unified approach for inversion problem in intensity-modulated radiation therapy.Phys. Med. Biol., 51 , 2353-2365

[4] Censor, Y. and Elfving, T.,(1994). A multiprojection algorithm using Bregman projection in a product space. Numer. Algorithms, 8 , 221-239

[5] Censor, Y., Elfving, T., Kopf, N. and Bortfeld, T.,(2005). The multi-set split feasibility problem and its applications. Inverse Problem, 21 , 2071-2084

[6] Censor, Y., Motova A. and Segal, A.,(2007). Pertured projections and subgradient projections for the multiple-sets split feasibility problem. J. Math. Anal. Appl., 327 , 1244-1256

[7] Censor, Y. and Segal, A.,(2009). The split common fixed point problem for directed operators. J. Convex Anal., 16 , 587-600

[8] Chang, S. S., Cho, Y. J., Kim, J. K., Zhang, W. B. and Yang, L.,(2012). Multiple-set split feasibility problems for asymptotically strict pseudocontractions. Abstr. Appl. Anal., Article ID 491760

[9] Chang, S. S., Joseph Lee, H. W., Chang, C. K., Wang, L . and Qin, L. J.,(2013). Split feasibility problem for Quasi-nonexpansive multi-valued mappings and total asymptotically strict pseudo contractive mappings. Applied Math. and Comp., 219 , 10416-10424

[10] Chang, S. S., Wang, L., Tang, Y. K. and Yang, L.,(2012). The split common fixed point problem for total asymptotically strictly pseudocontractive mappings. J. Appl. Math., Article ID 385638

[11] Chidume, C.E., Bello, A.U., Ndambonve, P.(2014). Δ-Convergence theorems for common fixed points of a finite family of multivalued demi-contractive mapping in CAT(0) spaces. Abstract Application Analysis, Article ID 805168.

[12] Chidume, C. E., Bello, A. U. and Onyido, M.A.,(2015). Convergence Theorem for a countable family of Multi-valued strictly pseudocontractive mappings in Hilbert spaces. Int'l journal of Math. Anal., 9, 1331-1340

[13] Chidume, C. E. and Ezeora, J. N., Krasnoselskii-type algorithm for family of multi-valued strictly pseudo-contractive mappings, Fixed Point Theory and Appl., 2014(2014):111

[14] Ezeora, J.N and Ogbonna, R.C.(2018). Split feasibility problem for countable family of multi-valued Nonlinear mappings. Mathematicki Vesnik., 70, 3,233-242.

[15] Hicks, T. L. and Kubicek, J.D.,(1977). On the Mann iteration process in Hilbert spaces. J.Math.Anal.Appl.,59 , 498 − 504.

[16] Masad, E. and Reich, S.,(2007). A note on the multiple-set split feasibility problem in Hilbert spaces. J.Nonlinear Convex Anal., 8 , 367-371

[17] Quan, J., Chang, S. S. and Zhang, X.,(2013). Multiple-set split feasibility problems for k-strictly pseudononspreading mapping in Hilbert spaces.Abstr. Appl. Anal., Article ID 342545 .

[18] Xu, H. K.,(2010). Iterative methods for split feasibility problem in infinite dimensional Hilbert spaces. Inverse problem, 26, 105018(17PP).

[19] Xu, H. K.,(2006). A variable Krasnosel'skii-Mann algorithm and the multiple-sets split feasibility problem. Inverse Problem, 22 , 2021-2034.

[20] Yang, L., Chang, S. S., Cho, Y. J. and Kim, J. K., Multiple-setsplit feasibility problems for total asymptotically strict pseudo contractions mappings, Fixed Point Theory Appl.

[21] Zhou, H.,(2008). Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces. Nonlinear Analysis, 69 , 3160-3173