A Study of Different Types of Ideals and The Propositions That Connect Them in PS-Algebra

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Abstract:

The concept of ideals in algebraic structures has undergone significant generalizations, especially within BCK, BCI, BCH, KU-algebra, to accommodate non-associative and non-commutative operations. While many types of PS-ideals have been studied, However, the extensive study of ideals in PS- algebra is still limited. This presents a notable gap in the literature, particularly regarding their formal characterization and existence in finite systems. This paper introduces and classifies distinct types of PS-ideals and focuses on the construction and verification of a non-trivial ideal within a finite set under a specifically defined binary operation. Through detailed analysis, we show that the subset ideal satisfies some conditions, offering a concrete example that enhances our understanding of ideal theory in generalized algebraic structures. The findings contribute to the theoretical development of PS-ideals and open new directions for further algebraic investigations, particularly in the study of minimal and non-classical algebraic systems.

Keywords: PS-algebra, PS-subalgebra, PS-ideal,PS-strong ideal,PS-closed ideal,PS-L-R-ideal,PS-n-m-ideal.

1-Introduction

In recent years, the study of algebraic structures has expanded significantly beyond traditional frameworks such as groups, rings, and fields, toward more generalized systems capable of modeling non-associative, non-commutative, and partial binary operations. This shift has resulted in the emergence of several non-classical algebras, including BCK-algebras, BCI-algebras, BCH-algebras, KU-algebras, and PS-algebras. Each of these structures supports various forms of ideal theory, which serve as a foundation for analyzing their internal behavior, equivalence relations, and decomposition properties [1][2][3].BCK-algebras and their generalizations, such as BCI- and BCH-algebras, originated in the study of logical systems and weak implication operations. These algebras have been extensively investigated not only in their crisp (classical) form but also in fuzzy, neutrosophic, and hybrid extensions [4][5][6]. The introduction of fuzzy ideals, bipolar ideals, and multipolar fuzzy ideals in these structures has broadened the algebraic understanding of uncertainty and approximation. Studies such as [7], [8], and [9] demonstrate the diverse applications and classifications of ideals in BCK-related algebras, including topological and cubic set frameworks. Similarly, KU-algebras, which provide a generalized framework for dealing with logic-based systems and soft computing, have seen considerable attention in recent literature. Several types of ideals have been introduced in KU-algebras, including fuzzy KU-ideals, positive implicative ideals, and multipolar fuzzy ideals. These ideals have been studied for their algebraic properties and their relationships with classical logical systems and semigroups [10][11][12].Parallel to these developments, PS-algebra have emerged as class of algebraic systems, where the binary operation may be partially defined or lack associativity. Despite their flexibility, PS-algebra have remained limited compared to BCK or KU-algebras. One of the most interesting aspects of PS-algebra is the diversity of their ideals: distinct types of ideals have been proposed, including, strong, closed, L, R, L-R, prime,n,n-m,multiple and others [13]. These definitions allow for more nuanced distinctions between elements and subsets within PS-algebras, offering a flexible setting for structural analysis.However, despite the abundance of ideal types defined in PS-structures, several of ideals remain understudied, especially in the context of finite PS-algebras. Unlike in BCK or KU-algebras, where closure-type and absorbing-type ideals have received extensive attention (e.g., fuzzy and cubic positive ideals [7][10]), ideals in PS-structures have rarely been examined through concrete finite examples. This presents a clear gap in the existing literature. In light of this, the present work focuses on the structure and properties of PS-ideals.The paper provides a formal classification of PS-ideal types, and focused ideal in a finite PS-structure defined over the finite set. Through this approach, we aim to enrich the theory of ideals in non-classical algebras and contribute to the broader understanding of general algebraic systems.

2- Preliminaries:

In this section we recall some fundamental definitions that will be used in the sequel

Definition 2.1 [14]: A PS-algebra is an al algebra$(P,* ,e)$ satisfying the following condtions:

$i)\ p_1 * p_1 = e$

$ii)\ p_1 * e = e$

$iii)\ p_1 * p_2 = e \text{ and } p_2 * p_1 = e \Rightarrow p_1 = p_2, \quad \forall\, p_1, p_2 \in P.$

Definition 2.2 [14]: A nonempty set S in PS-algebra $(P,*,e)$ is called PS-subalgebra if it is: $p_1 * p_2 \in S, \forall\, p_1, p_2 \in S$.

Definition 2.3 [14]: *Let* $(P,*,e)$ be a PS-algebra and I be a subset of P, then I is called a PS-ideal of P if it satisfies following conditions:

$i) e \in I$

$ii)\ p_2 * p_1 \in I \text{ and } p_2 \in I \Rightarrow p_1 \in I.$

3-The Facts and Results

Definition 3.1: A subset I in PS-algebra$(P,*,e)$ is called PS-strong ideal of P if it holds conditions:

$i)\ e \in I$

$.ii)\ p_1 * p_2 \in I, \forall p_1 \in I \Longrightarrow p_2 \in I$

Example 3.2: Let $P = \{e, p_1, p_2\}$ with following table *1:*

Table 1. $(P,*,e)$ **PS-algebra**

$*$	e	p_1	p_2
e	e	p_1	p_2
p_1	e	e	p_1
p_2	e	p_2	e

It is clear that $(P,*,e)$ PS-algebra , $I = \{e, p_1\}$ is PS-strong ideal.

Proposition 3.3: Every PS-ideal in PS-algebra is PS-strong ideal.

Proof: It is directly.

Remark 3.4: The converse of Proposition 3.3 is not true in general, from example3.2

$I = \{e, p_1\}$ is PS-strong ideal but not PS-ideal since $p_1 * p_2 = p_1 \in I \text{ and } p_1 \in I$

$but\ p_2 \notin I.$

Proposition 3.5: Let S be a PS-subalgebra contains e in PS-algebra (P , $*$, e), then S is a PS-strong ideal

Proof:

Assume S a PS-subalgebra and $e \in S$.

This implies $p_1 * p_2 \in S, \forall\, p_1, p_2 \in S$.

To prove S is PS-strong ideal,

Suppose $p_1 \in S$ & $p_1 * p_2 \in S$.

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Now, we show $p_2 \in S$

Since $p_1 \in S$ & $p_1 * p_2 \in S$, and also $p_1 * p_1 = e \in S$

If $p_1 * p_2 = e$ *and* $p_2 * p_1 = e$,from condition iii in Definition 2.1,

$p_1 = p_2 \Rightarrow p_2 \in S$

Hence, I is a PS-strong ideal

Remark 3.6: The opposite of Proposition 3.5 is not true.

Definition 3.7: A nonempty subset I of PS-algebra (P ,∗ , e) is called PS-closed ideal if satisfied the conditions:

$i)$ *If* $p \in I$ *then* $e * p \in I$

$ii) p_2 * p_1 \in I$ *and* $p_2 \in I \Longrightarrow p_1 \in I$.

Example 3.8: Let $P = \{e, x, y, z\}$ with binary operation ∗ in the following table 2:

Table 2. $(\boldsymbol{P}, *, \boldsymbol{e})$ **PS-algebra**

*	$\boldsymbol{e}$	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$
$\boldsymbol{e}$	e	x	y	z
$\boldsymbol{x}$	e	e	y	z
$\boldsymbol{y}$	e	y	e	y
$\boldsymbol{z}$	e	x	y	e

It is easy to show that *(P , ∗,e)* is PS-algebra, $I = \{e, x, z\}$ is PS-closed ideal.

Proposition 3.9: Every PS-closed ideal in PS-algebra is PS-ideal.

Proof: Let I be PS-closed ideal, this implies satisfies ii conditions in Definition 2.3.

Now, we prove $e \in I$ *, since for any* $p \in I \Longrightarrow e * p \in I$,

In particular,$p = e, e = e * e \in I$.

Therefore, I is a PS-ideal.

Corollary 3.10: Every PS-closed ideal is PS-strong ideal.

Proof: Frome Proposition 3.9 and Proposition 3.3.

Definition 3.11: Let (P ,∗ , e) be a PS-algebra. A subset I in P is called PS-L-ideal of P if it satisfies the following conditions:

$i) e \in I$

$ii), \forall p_1 \in P, p_2 \in I \Rightarrow p_1 * p_2 \in I$

Example *3.12:* Let $P = \{0,1,2\}$ and ∗ binary operation defined in the table 3:

Table 3. $(\boldsymbol{P}, *, \boldsymbol{0})$ **PS-algebra**

*	**0**	**1**	**2**
0	0	1	2
1	0	0	2
2	0	1	0

It is clear *(P , ∗, 0)* is PS-algebra ,and $I = \{0,2\}$ *is PS-L- ideal*

Since $0 \in I$,

Now,

If $p_1 = 0: 0 * 0 = 0 \in I, 0 * 2 = 2 \in I$

If $p_1 = 1: 1 * 0 = 0 \in I, 1 * 2 = 2 \in I$

If $p_1 = 2: 2 * 0 = 0 \in I, 2 * 2 = 0 \in I$

Therefore, I it is a PS-L- ideal.

Proposition 3.13: In PS-algebra *(P, ∗, e)* every PS-L- ideal is PS-strong ideal.

Proof: Suppose I is a PS-L- ideal in *P*

Let $p_1 \in I, \& \ p_1 * p_2 \in I.$

Since *I is a PS-L-* ideal,$\forall p \in P, p * p_1 \in I,$

In particular $p_2 * p_1 \in I.$

From condition iii in Definition 2.1 if $p_1 * p_2 = e \& p_2 * p_1 = e \Rightarrow p_1 = p_2$ *,thus* $p_2 \in I.$

In the others hand, if $p_2 \notin I$,it contradicts closure, so $p_2 \in I.$

Therefore, *I is a PS*-strong ideal.

Remark 3.14: The converse of Proposition 3.13 is not true.

Definition 3.15: A subset I in PS-algebra (P , ∗ , e) is called PS-R- ideal if:

$i) e \in I$

$ii) \forall \ p_2 \in I, p_1 \in P \Rightarrow p_2 * p_1 \in I.$

Example 3.16: Let $P = \{x, y, z\}$ with following table 4:

Table 4 . $(P,*,x)$ PS-algebra

$*$	x	y	z
x	x	x	x
y	x	x	z
z	x	x	x

(P, ∗, x) is a PS-algebra,

Now we check $I = \{x, z\}$ *is PS-R-* ideal.

,$x \in I$

$z * x = x \in I, z * y = x \in I, z * z = x \in I.$

Hence ,*I is a PS-R-* ideal.

Definition 3.17: Let $(P ,* \ , e)$ be a PS-algebra and I a subset in P, then I is called PS-L-R- ideal if:

$i) e \in I$

$ii) \forall p_2 \in I \& \ p_1 \in P \Rightarrow p_1 * p_2 \& \ p_2 * p_1 \in I.$

Example 3.18: Let P $= \{ e, x, y, z\}$ and operation ∗ defined in table5:

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Table 5. $(P,*,e)$ **PS-algebra**

$*$	e	x	y	z
e	e	x	y	z
x	e	e	x	x
y	e	y	e	y
z	e	x	y	e

It is easy to show $(P, *, e)$ is a PS-algebra.

We choose $I = \{e, x, y\}$,

Now, check *I is a PS-L-R-* ideal

$e \in I$

If $e \in P$:$e * e = e \in I, e * x = x \in I, e * y = y \in I$

$x \in P: x * e = e \in I, x * x = e \in I, x * y = x \in I$

$y \in P: y * e = e \in I, y * x = y \in I$

$z \in P: z * e = e \in I, z * x = x \in I, z * y = y \in I$

Hence, I is a PS-L-R- ideal

Corollary 3.19: Every PS-L-R- ideal in PS-algebra is PS-L- ideal.

Corollary 3.20: Every PS-L-R-ideal in PS-algebra is PS-strong ideal.

Corollary 3.21: Every PS-L-R-ideal in PS-algebra is PS-R- ideal.

Definition 3.22: A subset I in PS-algebra (P ,∗ , e) is called PS-prime ideal if satisfying the following conditions for all $p_1, p_2 \in P$.

$i) e \in I$

$ii)$ *If* $p_1 * p_2 \in I \Rightarrow p_1$ *or* $p_2 \in I$ such that $p_1 \neq p_2$.

Example 3.23: Take the set $P = \{1,2,3,4\}$ with following table 6 *:*

Table 6. $(P,*,1)$ **PS-algebra**

$*$	**1**	**2**	**3**	**4**
1	1	2	3	4
2	1	1	2	2
3	1	3	1	3
4	1	2	3	1

$(P, *, 1)$ it is a PS-algebra from table,

We check $I = \{1,2\}$

$1 * 2 = 2 \in I, 1,2 \in I, 2 * 1 = 1 \in I, 1,2 \in I$

$2 * 3 = 2 \in I, 2 \in I, 2 * 4 = 2 \in I, 2 \in I$

$4 * 1 = 1 \in I, 1 \in I, 4 * 2 = 2 \in I, 2 \in I.$

Hence, *I is a PS*-prime ideal

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Corollary 3.24: Every PS-prime ideal in PS-algebra is a PS-ideal.

Corollary 3.25: Every PS-prime ideal in PS-algebra is a PS-strong ideal.

Definition 3.26: Let $(P,*,e)$ be PS-algebra ,a subset I is called PS-n- ideal of P if it is satisified:

i)$e \in I$.

ii)$p^n = e, \forall p \in I, for\ some\ n \in \mathbb{N}, n \geq 3$.

Example3.27: Consider $P = \{e, p_1, p_2\}$,and operation $*$ define on following table7:

Table 7. $(P,*,e)$ PS-algebra

$*$	e	p_1	p_2
e	e	e	e
p_1	e	e	p_1
p_2	e	e	e

It is clear that *(P, $*$,e)* is PS-algebra,

Choose $I = \{e, p_2\}$

$e^3 = e$

${p^3}_2 = p_2 * p_2 * p_2 = e$

Therefore, *I is a PS-n- ideal.*

Definition 3.28: Let (P , $*$, e) be a PS-algebra then I is called PS- n-m- ideal if:

For every $p \in P$, *i*f there exists $n \geq 1$ such that $p^n \in I$,then for all $m \geq n$,we also have $p^m \in I$.
Example 3.29: Let $P = \{1,2,3,4\}$ with the binary operation $*$ defined by table 8:

Table 8. $(P,*,1)$ PS-algebra

*	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	1	4	1
3	1	4	1	1
4	1	1	1	1

From table *(P,$*$, 1)* is a PS-algebra

We choose $I = \{1,4\}$

Check I the PS-n-m ideal:

$2^2 = 1 \in I \Rightarrow 2^3 = 2^2 * 2^1 = 1 * 2 = 1 \in I$

$3^2 = 1 \in I \Rightarrow 3^3 = 3^2 * 3 = 1 * 3 = 1 \in I$

$4^2 = 1 \in I \Rightarrow 4^3 = 4^2 * 4 = 1 * 4 = 1 \in I$

Therefore, I is a PS-n-m ideal.

Proposition 3.30: If I is a PS-n-m- ideal in PS-algebra (P , $*$, e), then I is a PS-n-ideal.

Proof: Directly

Remark 3.31: The converse of Proposition 3.30 is not true always the following example explain that:

Example 3.32: Let $P = \{e, x, y, z\}$ with following table 9:

Table 9. $(P,*,e)$ **PS-algebra**

$*$	e	x	y	z
e	e	e	x	e
x	e	e	z	e
y	e	z	e	e
z	e	e	e	e

(P , $*$, e) is a PS-algebra

Choose $I = \{e, y\}$ is PS-n-ideal but not PS-n-m ideal

Since $, y \in I$, *but* $y^3 = y * y * y = e * y = x \notin I.$

4- Conclusion

This study explored various types of ideals within PS-algebras, providing a comprehensive theoretical framework to clarify their structural roles and internal interactions.The analysis included a wide range of ideals such as PS-strong ideal, PS-closed ideal,PS-L-ideal,PS-R-ideal,PS-L-R-ideal,PS-n-m-ideal and PS-n-ideal.Each of these contributes uniquely to understanding more properties within non classical algebraic systems.The significance of these ideals extends beyond PS-algebras, offering a foundation for conceptual expansion into more generalized algebraic frameworks. The diversity of these ideal structures presents powerful tools for analyzing complex algebraic systems and supports a variety of applications across logic,computational theory.Therefore,the development and continued investigation of these ideals represent a critical step toward advancing modern algebra and its practical implementations across interdisciplinary fields.

Conflicts Of Interest

The author declare no conflicts of interest.

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